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Kinematical Groups as Group Extensions

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Abstract

Classical kinematical groups are analysed from the viewpoint of the cohomology theory of abstract groups using group extension and G-enlargement theory.

Introduction

The purpose of this article is to discuss the kinematical groups of physics from the point of view of group extension theory. A group extension may be loosely described as a group which is obtained by weaving together two other groups. Most of the kinematical groups of physics are in fact group extensions. The best known and most thoroughly investigated of these is the Poincaré group of special relativity. The Poincaré group is an example of the most simple kind, a semi-direct product. Its structure may be symbolised by (Michel, 1967)

$$P = ST X n L$$

where ST is the spatiotemporal translation group, L the Lorentz group and the symbol 'n' denotes the natural action of L as a group of automorphisms of ST. The notation in brief means that $P = ST.L$, $ST \lhd P$, $L < P$, $P/ST = L$, $ST \cap L = 1$ and for $t \in ST$, $\Lambda \in L$; $\Lambda t \Lambda^{-1} = \Lambda . t$ and specifies P completely in terms of ST and L. P is said to be a semi-direct product of L by ST.

The Galilei group of Newtonian relativity (Lévy-Leblond, 1971; Whiston, 1972) has a richer structure than the Poincaré group, in that it has a spectrum of group extension structures. If we denote by H the homogeneous Galilei group, by T the temporal translation group and by S the spatial translation group, then the Galilei group G may also be written as a semi-direct product

$$G \cong (SXT) X\rho H$$

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Where $\rho \in \mathrm{Hom}(H, \mathrm{Aut}(SXT))$ denotes the action of H as a group of automorphisms of SXT in this extension and 'X' denotes direct product. In fact ρ is given by:

$$\rho(v,r) : (x,t) \mapsto (r.x + vt, t)$$

for $(v,r) \in H$; $(x,t) \in SXT$ (note that H is itself a semi-direct product of the rotation group by H_0 the abelian group of pure inertial boosts). This extension structure is analogous to that of P in that the spatiotemporal translations $ST \cong SXT$ are an invariant subgroup of G and the quotient of G by the translations is a rotation-inertial boost group. However, as we have mentioned, the Galilei group has a spectrum of other group extension structures. Amongst these is the following, the most interesting group theoretically:

$$G \cong S \otimes_{\xi} (TXH)$$

which is a non-trivial group extension in that TXH is not a subgroup. The object 'ξ' measures the deviation of TXH from being a subgroup. This notation will be better described in the next section. The above structure of the Galilean group, apart from the non-primitive space groups of crystallography (Ascher & Janner, 1968), is perhaps the only example of a non-split group extension occurring naturally in physics.

Recently (Lévy-Leblond, 1965), in an analysis of the Galilei group as a 'non relativistic limit' of the Poincaré group, Lévy-Leblond discovered a non-physical kinematical group, the Carroll group, which is another degenerate cousin of the Poincaré group. The Carroll group holds more group theoretic interest than physical, but it has an analogous spectrum of extension structures to the Galilean group, in which the formal roles of space and time are interchanged. One may either write the Carroll group C as

$$C \cong (SXT)\, X\rho'\, H$$

where $\rho' \in \mathrm{Hom}(H, \mathrm{Aut}(SXT))$ is defined by the rule

$$\rho'(v,r) : (x,t) \mapsto (r.x, t + v.r.x)$$

for $(v,r) \in H$, and $(x,t) \in SXT$. It has a non-split structure too, which we write as:

$$C \cong T \otimes_{\xi'} (SX_{\pi'} H)$$

H acting as a group of rotations on S. Another kinematical group of slight physical interest is the so-called 'static' group $\tilde{S}$ (Lévy-Leblond & Bacry, 1968) which has a unique trivial structure

$$\tilde{S} \cong (SXT)\, X\rho''\, H$$

where this time $\rho'' \in \mathrm{Hom}(H, \mathrm{Aut}(SXT))$ is defined by

$$\rho''(v,r) : (x,t) \mapsto (r.x, t)$$

Hence the title 'static' as the inertial boosts have a trivial action on SXT.

Let us reconsider these three kinematical groups together. If we look at their semi-direct product forms, we notice that they differ only in the actions of H as a group of automorphisms of SXT:

$$p(v,r):(x,t)\mapsto(r.x+vt,t) \qquad \text{(Galilei)}$$

$$p'(v,r):(x,t)\mapsto(rx,t+v.r.x) \qquad \text{(Carroll)}$$

$$p''(v,r):(x,t)\mapsto(r.x,t) \qquad \text{(Static)}$$

Up to rotations ρ leaves T 'invariant', ρ' leaves S 'invariant' and ρ'' leaves both 'invariant'. We shall formally call any group extension a kinematical group if it is a split extension of H by SXT in which H acts internally in the usual way on S and T (that is trivially on T and as a group of rotations in S) and investigate the consequences. In the course of this investigation, we shall show how the structures of the Galilei type (which we call T-kinematical groups) and of the Carroll type (which we call S-kinematical groups) induce non-equivalent non-split extension structures and we shall attempt to compute *all* S and *all* T-kinematical groups, solving the problem in principle, if perhaps not in fact. We shall also explain exactly what we mean by S or T-kinematical groups below.

Mathematical Apparatus

A group G is said to operate on a group A (Scott, 1964) if there is an action of G as a group of automorphisms of A, that is, a homomorphism $p\in \mathrm{Hom}(G,\mathrm{Aut}(A))$. If this is the case we call A a G-group. (Note that any group is a G-group since G can act trivially on any group.) If A_1 and A_2 are G-groups, a G-homomorphism from A_1 into A_2 is a homomorphism $f\in \mathrm{Hom}(A_1,A_2)$ such that $f(g.\alpha)=g.(f(\alpha))$ for any $g\in G$, $\alpha\in A$. We shall write $p(g)(\alpha)$ in the form $g.\alpha$ when no confusion can arise. If f is a G-homomorphism we shall write $f\in \mathrm{Hom}_G(A_1,A_2)$. If R is a ring, an abelian group A is called an R-module if there is a function $k:RXA\to A$; $k:(r,\alpha)\mapsto r.\alpha$ such that if 1, 0 are the identities of R, $1.\alpha=\alpha$; $0.\alpha=0$ and

$$(r_1r_2).\alpha=r_1.(r_2.\alpha), \qquad r.(\alpha_1+\alpha_2)=r.\alpha_1+r.\alpha_2$$

and

$$(r_1+r_2).\alpha=r_1.\alpha+r_2.\alpha$$

Now if A is an abelian G-group it is an R-module for R the group-ring (Maclane, 1963) of G. Thus we call an abelian G-group a G-module.

Suppose that K, an abelian group is a Q-module for some group Q, defined as such via an action $p\in \mathrm{Hom}(Q,\mathrm{Aut}(K))$. Let $C_p{}^n(Q,K)$ denote all functions from Q^n to K for $n\geqslant 1$, and for $n=0$, let $C_p{}^0(Q,K)=K$; $C_p{}^n(Q,K)$ are all Q-modules in the obvious way. There is a complex $Cp(Q,K)$ whose nth components are the $C_p{}^n(Q,K)$ for $n\geqslant 0$ and trivial modules for $n < 0$. The coboundary homomorphisms $\delta^n \in \mathrm{Hom}(C_p^n(Q,K), C_p^{n+1}(Q,K))$ are defined by $\delta^n = 0$ for $n < 0$ and for $n \geq 0$

$$\delta^n(f)(q_1 \ldots q_{n+1}) = (-1)^{n+1} f(q_1 \ldots q_n) + q_1 . f(q_2 \ldots q_n)$$
$$+ \sum_{i=1}^{n} (-1)^i f(q_1 \ldots q_i q_{i+1} \ldots q_{n+1})$$

One can show that $\delta^{n+1} \circ \delta^n = 0$. The groups $Z_p^n(Q,K)$ and $B_p^n(Q,K)$ are defined to be respectively $\mathrm{Ker}(\delta^n)$ and $\mathrm{Im}(\delta^{n-1})$; because $B_p^n(Q,K) < Z_p^n(Q,K)$ we may define a group $H_p^n(Q,K)$ as $Z_p^n(Q,K)/B_p^n(Q,K)$. The group $Z_p^n(Q,K)$ is called the group of n-cocycles, $B_p^n(Q,K)$ the group of n-coboundaries and $H_p^n(Q,K)$ the n-dimensional cohomology of Q in K. If $z \in Z_p^n(Q,K)$ is an n-cocycle of Q in K we shall denote $|z|$ for the 'cohomology class of z' in $H_p^n(Q,K)$. Suppose K is an R-module for some ring R and that for all $q \in Q, p(q)(r.k) = r.p(q)(k)$, then (because $C_p^n(Q,K)$ are also R-modules), the coboundary homomorphisms are R-homomorphisms $Z_p^n(Q,K)$, $B_p^n(Q,K)$ are R-modules and thus so are $H_p^n(Q,K)$.

A group E is said to be a group extension of a group Q by a group K iff E fits in a short exact sequence:

$$K \overset{}{\underset{i}{\rightarrowtail}} E \underset{\phi}{\twoheadrightarrow} Q$$

This diagram is shorthand for i is a monomorphism ($\rightarrowtail$) ϕ is an epimorphism ($\twoheadrightarrow$) and $\mathrm{Ker}(\phi) = \mathrm{Im}(i)$. Consequently $i: K \lhd E$, $E/i(K) = Q$. E is said to 'split' on the right iff there is a homomorphism $j \in \mathrm{Hom}(Q,E)$ such that $\phi \circ j = 1_Q$, in which case j is a monomorphism. We shall call an extension of this type a semi-direct product. An extension is called split on the left iff there is an epimorphism $j \in \mathrm{Hom}(E,K)$ such that $j \circ i = 1_K$. A doubly split extension is called a direct product. If E is a semi-direct product of Q by K, the inner automorphisms of E through Q induce an action of Q on K as a group of automorphisms, that is, a homomorphism $g \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ such that, because $E = i(K).j(Q)$, the group law on E is given by, $(g(q)(k) \equiv q.k)$

$$(i(k_1)\,j(q_1))\,(i(k)\,j(q_2)) = (i(k_1 q_1 . k_2)\,j(q_1 q_2))$$

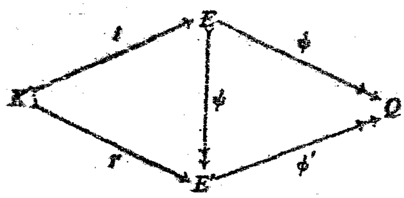
Conversely, given $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ we may construct a group on the set KXQ with multiplication table

$$(k_1, q_1).(k_2, q_2) = (k_1 q_1 . k_2, q_1 q_2)$$

and this group is a semi-direct product of Q by K. Note that the semi-direct product with trivial action is the direct product. If E is a semi-direct product of Q by K and $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ defines K as a Q-group we shall denote the group by

$$E = KX_p Q$$

The usual notation $K \times Q$ will be used for the direct product. Two group extensions are said to be equivalent iff there is an isomorphism $\psi : E \cong E'$ such that the diagram below commutes



Suppose now that K is an abelian group and that E is a group extension of Q by K. If s is any section of ϕ over Q there is an action $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ of Q on K and a function $w_s \in C_p^2(Q, K)$ such that the group law on E is

$$(i(k_1)s(q_1))(i(k_2)s(q_2)) = i(k_1 + q_1 . k_2 + w_s(q_1, q_2))s(q_1 q_2)$$

where in fact w_s is defined by

$$i(w_s(q_1, q_2)) = s(q_1)s(q_2)(s(q_1 q_2))^{-1}$$

Now although the action p of Q on K is independent of the choice of section s (because K is abelian) the above group law does depend on s through w_s which is a cocycle of

$$Z_p^2(Q, K)$$

which is a requirement of associativity in E. If we make a new choice s of section of ϕ over Q, the cocycles w_s and $w_{s'}$ are related by $w_{s'} = w_s + \delta(f)$, where $f \in C_p^1(Q, K)$ is defined by:

$$i(f(q)) = s'(q)s(q)^{-1}$$

Thus a different extension is obtained. However, the two extensions are equivalent in that the isomorphism

$$I : E \to E'$$

$$I : i(k)s(q) \mapsto i(k + f(q))s'(q)$$

makes the aforementioned diagram commute. If E and E' are equivalent extensions of Q by K the cocycles which determine the group multiplication tables of E and E' are cohomologous. For if $\sigma : E \cong E'$ is an isomorphism realising the equivalence

$$\sigma(s(q_1)s(q_2)) = \sigma(s(q_1))\sigma(s(q_2)) = s'(q_1)s'(q_2)$$

where s' is a section of ϕ' over Q and so

$$\sigma(i(w_s(q_1, q_2))s'(q_1 q_2) = i'(w_{s'}(q_1, q_2))s'(q_1 q_2)$$

implying $w_s(q_1; q_2) = w_{s'}(q_1, q_2)$ since $i' = \sigma \circ i$ and i' is a monomorphism. But $w_{s'}$ differs from the original factor system of E' by a coboundary, hence w_s must be cohomologous to the original. We may therefore define a function

$$w : \text{Ext}_p(Q, K) \to H_p^2(Q, K)$$
$$w : |E| \quad\quad \mapsto |w_s|$$

for any section of ϕ over Q.

$\text{Ext}_p(Q, K)$ is the set of equivalence classes $|E|$ of extensions of Q by K. That w is a bijection is shown by noting that given a two cocycle $\xi \in Z_p^2(Q, K)$ the group on KXQ with multiplication table

$$(k_1, q_1).(k_2, q_2) = (k_1 + q_1 . k_2 + \xi(q_1, q_2), q_1 q_2)$$

E_ξ, is a group extension of Q by K and that if $|\xi| = |\xi'|$ then $|E_\xi| = |E_{\xi'}|$. The inverse to w is thus $E : |\xi| \mapsto |E_\xi|$ for any $\xi \in Z_p^2(Q, K)$. Thus we see that there is a 1–1 correspondence $\text{Ext}_p(Q, K) \cong H_p^2(Q, K)$. This correspondence can be extended to a group isomorphism using the Baer product (Maclane, 1963) of group extensions. For any cocycle $\xi \in Z_p^2(Q, K)$ we shall denote the group extension determined up to equivalence by ξ as

$$K \otimes_\xi Q$$

Suppose that E is a split extension of Q by an abelian group K which is a trivial Q-module and that Q and K are G-groups from some group G, defined as such via

$$P_K \in \text{Hom}(G, \text{Aut}(K)), \qquad p_Q \in \text{Hom}(G, \text{Aut}(Q))$$

E will be called a G-enlargement (Eilenberg, 1949) of Q by K iff (i) E is a G-group via $p_E \in \text{Hom}(G, \text{Aut}(E))$, (ii) $\phi : E \twoheadrightarrow Q$ is in $\text{Hom}_G(E, Q)$ and (iii) $i : K \lhd E$ is in $\text{Hom}_G(K, E)$. Note that the splitting homomorphism of Q into E is not necessarily supposed to be a G-homomorphism. Two G-enlargements of Q by K are called G-equivalent iff their split extensions are equivalent through a G-isomorphism. Essentially then, a G-enlargement E of Q by K is the giving of an action P_E of G in KXQ which extends the actions of G in K and Q. Note also that the split extension of Q by K are in correspondence with the family of splitting homomorphisms of Q into KXQ. The 'factoring out' of this arbitrariness is attained by the notion of G-equivalence as we shall see in the proof of the following proposition.

Proposition (1) (Eilenberg, 1949): If $\text{Ext}_G(Q, K)$ is the set of G-equivalence classes of G-enlargements of Q by K, there is a 1–1 correspondence:

$$\text{Ext}_G(Q, K) \cong H_p^1(G, \text{Hom}(Q, K))$$

The action of G in $\text{Hom}(Q, K)$ being defined by $P(g)(f) = P_K(g) \circ f \circ P_Q(g^{-1})$ for any $f \in \text{Hom}(Q, K)$, $g \in G$.

Proof: Suppose that E is a G-enlargement of Q by K defined as such via an action $P_E \in \mathrm{Hom}(G, \mathrm{Aut}(E))$. Let j be a monomorphic section of $\phi : E \twoheadrightarrow Q$ over Q, not necessarily a G-homomorphism. Define the obstruction $w_j^{p_E}$ to j being a G-homomorphism by

$$w_j^{p_E} \in C_p^1(G, C(Q, K))$$

and

$$i(w_j^{p_E}(g)(q)) = g.(j(g^{-1}.q))\, j(q)^{-1}$$

We shall show that $w_j^{p_E} \in Z_p^1(G, \mathrm{Hom}(Q, K))$.

(1) $w_j^{p_E}(g) \in \mathrm{Hom}(Q, K)$ for any $g \in G$. For given $q_1, q_2 \in Q$

$$\begin{aligned} i(w_j^{p_E}(g)(q_1 q_2)) &= g.(j' g^{-1}.q_1 q_2)\, j(q_1 q_2)^{-1} \\ &= g.j(g^{-1}.q_1)\, g.j(g^{-1}.q_2)\, j(q_2)^{-1} j(q_1)^{-1} \\ &= g.j(g^{-1}.q_1)\, i(w_j^{p_E}(g)(q_2))\, j(q_1)^{-1} \\ &= i(w_j^{p_E}(g)(q_1))\, i(w_j^{p_E}(g)(q_e)) \\ &= i(w_j^{p_E}(g)(q_1) + w_j^{p_E}(g)(q_2)) \end{aligned}$$

Thus because i is a monomorphism we obtain

$$w_j^{p_E}(g)(q_1 q_2) = w_j^{p_E}(g)(q_1) + w_j^{p_E}(g)(q_2)$$

(2) $w_j^{p_E} \in Z_p^1(G, \mathrm{Hom}(Q, K))$. For if $g_1, g_2 \in G$, $q \in Q$ we have

$$\begin{aligned} i(w_j^{p_E}(g_1 g_2)(q)) &= (g_1 g_2.j((g_1 g_2)^{-1}.q))\, j(q)^{-1} \\ &= g_1.(g_2.j(g_2^{-1}.(g_1^{-1}.q))\, j(g_1^{-1}.q)^{-1})(g_1.j(g_1^{-1}.q)\, j(q)^{-1}) \\ &= g_1.i(w_j^{p_E}(g_2)(g_1^{-1}.q))\, i(w_j^{p_E}(g_1)(q)) \\ &= i(g_1.w_j^{p_E}(g_2)(g_1^{-1}.q)) + w_j^{p_E}(g_1)(q)) \end{aligned}$$

Therefore, we obtain the required result:

$$w_j^{p_E}(g_1 g_2) = w_j^{p_E}(g_1) + p(g_1)(w_j^{p_E}(g_2))$$

Suppose we make a new choice of section j' of over Q, keeping p_E fixed. Then because $j'(q) = i(\psi(q))\, j(q)$ for $\psi \in \mathrm{Hom}(Q, K)$, we obtain $w_{j'}^{p_E} = w_j^{p_E} + \delta(\psi)$ and hence $w^{p_E} \equiv |w_j^{p_E}|$ for any section j depends only on p_E. Now suppose E and E' are equivalent G-enlargements of Q by K. Then E and E' are equivalent expansions of Q by K and the isomorphism $\sigma : E \cong E'$ is a G-isomorphism. If j, j' are sections of ϕ, ϕ' we obtain. $\sigma : i(k)\, j(q) \mapsto i(k)\, j'(q)$ is a G-isomorphism and hence

$$\begin{aligned} \sigma(p_E(g)(j(q))) &= \sigma(i(w_j^{p_E}(g)(q))\, j(g.q)) \\ &= i(w_j^{p_E}(g)(q))\, \sigma(j(g.q)) \\ &= i(w_j^{p_E}(g)(q))\, j'(g.q) \\ &= p_{E'}(g)(j'(q)) \\ &= i(w_{j'}^{p_{E'}}(g)(q))\, j'(g.q) \end{aligned}$$

This implies $w_j^{p_E} = w_{j'}^{p_{E'}}$ and hence $w^{p_E} = w^{p_{E'}}$ and thus that the mapping $w : |E| \mapsto w^{p_E}$ is a function. It is a bijection because the mapping from

$H^1p(G,\mathrm{Hom}(Q,K))E: |w| \mapsto |E_w|$ where E_w is the G-enlargement defined on KXQ by $g.(k,q) = (g.k + w(g)(q), g.q)$ for any $w \in Z_p{}^1(G,\mathrm{Hom}(Q,K))$ is an inverse function. That E is a function follows from the assertion $|w| = |w'| \Rightarrow |E_w| = |E_{w'}|$.

Remark: One may define a composition of G-enlargements and show that using this composition, the bijection of proposition (1) is an isomorphism of abelian groups. We do not use this method but equivalently define an abelian group structure in $\mathrm{Ext}_G(Q,K)$ by 'pushing back' the addition in $H_p{}^1(G,\mathrm{Hom}(Q,K))$. The trivial enlargement is the identity for this addition in $\mathrm{Ext}_G(Q,K)$.

$$|E_w| + |E_{w'}| = |E_{w+w'}| \qquad \text{for } w, w' \in Z_p{}^1(G, \mathrm{Hom}(Q,K))$$

If K is an R-module for a ring R and $p_K(g)(r.k) = r.p_K(g)(k)$ for all $r \in R$, $g \in G$, $k \in K$, $H_p{}^1(G,\mathrm{Hom}(Q,K))$ is an R-module and hence by pushing back the R-module structure to $\mathrm{Ext}_G(Q,K)$ so is this group.

$$r.|E_w| = |E_{r.w}|$$

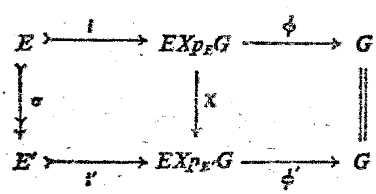
Thus when R is a field, $\mathrm{Ext}_G(Q,K)$ is a linear space over R.

Remark: If E is a G-enlargement of Q by K, there is a semi-direct product $EX_{p_E}G$ of G by KXQ. Let us call $T(p_K,p_Q)$ the trivial case in which $p(g)(i(k)\,j(q)) = i(p_K(g)(k))\,j(p_Q(g)(q))$, i.e. $T(p_K,p_Q) = E_0Xp_EG$, where 0 is the zero cocycle of $Z_p{}^1(G,\mathrm{Hom}(Q,K))$. We then call any semi-direct product $EX_{p_E}G$ of KXQ by G when E is a G-enlargement of Q by K a G-extension of $T(p_K,p_Q)$ and call any two G-extensions of $T(p_K,p_Q)$ equivalent iff they are equivalent as group extensions of G by KXQ.

Proposition (2): There is a 1–1 correspondence between equivalence classes of G-extensions of $T(p_K,p_Q)$ and the group $H_p{}^1(G,\mathrm{Hom}(Q,K))$.

Proof: There is a one to one correspondence between equivalence classes of G-enlargements of Q by K and equivalence classes of G-extensions of $T(p_K,p_Q)$. We demonstrate that the mapping $f: E \mapsto EXp_EG$ is a bijective function.

(1) f is a function, for if $|E| = |E'|$, $|EXp_EG| = |E'Xp_{E'}G|$. For if $|E| = |E'|$, E and E' are equivalent group extensions of Q by K and the isomorphism realising this equivalence is a G-homomorphism. Suppose that this is $\sigma: E \cong E'$ and define $\chi: EXp_EG \to E'Xp_{E'}G$ by $\chi: (\varepsilon,g) \mapsto (\sigma(\varepsilon),g)$ for $\varepsilon \in E$, $g \in G$. We claim that χ is a group isomorphism and that the diagram below commutes



where i, i' and ϕ, ϕ' are the homomorphisms

$$i, i' \varepsilon \mapsto (\varepsilon, e_G) \varepsilon \in E \text{ or } E', \phi, \phi' : (\varepsilon, g) \mapsto g, \varepsilon \in E \text{ or } E', g \in G$$

χ is a group homomorphism. For we have

$$\begin{aligned}\chi((\varepsilon_1, g_1)(\varepsilon_2, g_2)) &= \chi(\varepsilon_1 p_E(g)(\varepsilon_2), g_1 g_2) \\ &= (\sigma(\varepsilon_1 p_E(g)(\varepsilon_2)), g_1 g_2) \\ &= (\sigma(\varepsilon_1)\sigma(p_E(g)(\varepsilon_2)), g_1 g_2) \\ &= (\sigma(\varepsilon_1) p_{E'}(g)(\sigma(\varepsilon_2)), g_1 g_2) \\ &= (\sigma(\varepsilon_1), g_1)(\sigma(\varepsilon_2), g_2) \\ &= \chi(\varepsilon_1, g_1)\chi(\varepsilon_2, g)\end{aligned}$$

Similarly $\bar{\chi} : (\varepsilon, g) \mapsto (\sigma^{-1}(\varepsilon), g)$ for $\varepsilon \in E'$, $g \in G$ is a group homomorphism and it is the inverse to χ, so χ is a group isomorphism. The diagram obviously commutes so that $|EX_{p_E} G| = |E' X_{p_{E'}} G|$ and f is a function.

(2) The inverse to f is defined as the mapping $\bar{f}$ where

$$\bar{f} : |(KXQ) X_p G| \mapsto |E_p|$$

where $p_E = p$ for a G-extension of $T(p_K, p_Q)$. $\bar{f}$ is a function for, if the G-extensions $(KXQ) X_p G$, $(KXQ) X_{p'} G$ are equivalent, $|E_p| = |E_{p'}|$. Because $(KXQ) X_p G$ is equivalent as a group extension to $(KXQ) X_{p'} G$ there is a group isomorphism $\sigma : (KXQ) X_p G \cong (KXQ) X_{p'} G$. Hence

$$\sigma(p(g)(\varepsilon)) = \sigma(J(g)\varepsilon J(g)^{-1}) = \sigma(J(g))\varepsilon\sigma(J(g))^{-1} \equiv p'(g)(\varepsilon)$$

and σ induces a G-isomorphism $E \cong E'$ implying $|E_p| = |E_{p'}|$. It is clear that $\bar{f}$ is the inverse to f and hence f is a bijection.

Remark: We extend f to an isomorphism of abelian groups by pushing the group structure in $\mathrm{Ext}_G(Q, K)$ into the set of equivalence classes of G-extensions of $T(p_K, p_Q)$:

$$|(KXQ) X_p G| + |(KXQ) X_{p'} G| = |(KXQ) X_{p''} G|$$

where p'' is the action of G on the G-enlargement $E_p + E_{p'}$. If K is an R-module and $p_K(g)(r.k) = r.p(g)(k)$ for all $g \in G$, $r \in R$, $k \in K$, the set of equivalence classes of G-extensions of $T(p_K, p_Q)$ is an R-module with zero element $T(p_K, p_Q)$. Let us use the symbol $\mathrm{Ext}(T(p_K, p_Q))$ for this set of equivalence classes of G-extensions of $T(p_K, p_Q)$.

Corollary: There is an R-module isomorphism ($R = \mathbb{Z}$ in the trivial case) $H_p{}^1(G, \mathrm{Hom}(Q, K)) \cong \mathrm{Ext}(T(p_K, p_Q))$.

Proof: Combine propositions (1) and (2).

Remark: We now show how the semi-direct product structures of G-extensions of $T(p_K, p_Q)$ induce non-split extensions of $QXp_Q G$ by K.

***Proposition* (3):** There is a homomorphism

$$\gamma : H_p{}^1(G, \mathrm{Hom}(Q, K)) \to H^2_{\bar{p}K}(QXp_Q G, K)$$

Where, for $(q,g) \in QXp_Q G$, $\bar{p}_K(q,g) \equiv p_K(g)$.

***Proof*:** Define $\bar{\gamma} : Z_g{}^1(G, \mathrm{Hom}(Q, K)) \to Z^2_{\bar{p}K}(QXpG, K)$

$$\bar{\gamma}_\sigma((q_1, g_1), (q_2, g_2)) \equiv \sigma(g_1)(g_1 . q_2)$$

for

$$(q, g) \in QX_{p_Q} G \qquad \text{and} \qquad \sigma \in Z_p{}^1(G, \mathrm{Hom}(Q, K))$$

That $\bar{\gamma}_\sigma \in Z^2_{\bar{p}_K}(QXp_Q G, K)$ follows from the fact that

$$\begin{aligned}
\delta^2(\bar{\gamma}_\sigma)((q_1, g_1), (q_2, g_2), (q_3, g_3)) &= g_1 . \bar{\gamma}_\sigma((q_2, g_2), (q_3, g_3)) \\
&\quad - \bar{\gamma}_\sigma((q_1 g_1 . q_2, g_1 g_2), (q_3, g_3)) \\
&\quad + \bar{\gamma}_\sigma((q_1, g_1), (q_2 g_2 . q_3, g_2 g_3)) \\
&\quad - \bar{\gamma}_\sigma((q_1, g_1), (q_2, g_2)) \\
&= g_1 . \sigma(g_2)(g_2 . q_3) - \sigma(g_1 g_2)(g_1 g_2 . q_3) \\
&\quad + \sigma(g_1)(g_1 . (q_2 g_2 . q_3)) - \sigma(g_1)(g_1 . q_2) \\
&= g_1 . \sigma(g_e)(g_2 . q_3) - g_1 . \sigma(g_2)(g_1^{-1} . g_1 g_2) \\
&\quad - \sigma(g_1)(g_1 g_2 . q_3) + \sigma(g_1)(g_1 . q_2) \\
&\quad + \sigma(g_1)(g_1 g_2 . q_3) - \sigma(g_1(g_1 . q_2) \\
&= 0
\end{aligned}$$

Suppose that $\sigma_1, \sigma_2 \in Z_p{}^1(G, \mathrm{Hom}(Q, K))$, we show that $|\sigma_1| = |\sigma_2|$ implies $|\bar{\gamma}_{\sigma_1}| = |\bar{\gamma}_{\sigma_2}|$. For, we have

$$\begin{aligned}
\bar{\gamma}_{\sigma_1}((q_1; g_1), (q_e, g_2)) - \bar{\gamma}_{\sigma_2}((q_1, g_1), (q_2, g_2)) &= \sigma_1(g_1)(g_1 . q_2) - \sigma_2(g_1)(g_1 . q_2) \\
&= \delta(\psi)(g_1)(g_1 . q_2)
\end{aligned}$$

for $\psi \in \mathrm{Hom}(Q, K)$.

Now $\delta(\psi)(g_1)(g_1 . q_2) = g_1 . \psi(g_2) - \psi(g_1 . q_2)$. Let us define $\chi : QXp_Q G \to K$ by $(q,g) = \psi(q)$ for $q \in Q$, $g \in G$. Then

$$\begin{aligned}
\delta(\chi)((q_1, g_1), (q_2, g_2)) &= g_1 . \chi(q_2, q_2) - \chi(q_1 g_1 . q_2, g_1 g_2) + \chi(q_1, g_1) \\
&= g_1 . \psi(q_2) - \psi(q_1) - \psi(g_1 . q_2) + \psi(q_1) \\
&= g_1 . \psi(q_2) - \psi(g_1 . q_2)
\end{aligned}$$

Therefore $\bar{\gamma}_{\sigma_1} - \bar{\gamma}_{\sigma_2} = \delta(\chi)$ or $|\bar{\gamma}_{\sigma_1}| = |\bar{\gamma}_{\sigma_2}|$ and γ defined by $\gamma : |\sigma| \mapsto |\bar{\gamma}_\sigma|$ is a function from $H'p(G, \mathrm{Hom}(Q, K))$ into $H^2_{\bar{p}_K}(QXp_Q G, K)\gamma$ is in fact a group homomorphism. For $\bar{\gamma}$ is

$$\begin{aligned}
\bar{\gamma}_{\sigma_1 + \sigma_2}((q_1, g_1), (q_2, g_2)) &= \sigma_1 + \sigma_2(g_1)(g_1 . q_2) \\
&= (\sigma_1(g_1) + \sigma_2(g_2))(g_1 . q_2) \\
&= \sigma_1(g_1)(g_1 . g_2) + \sigma_2(g_1)(g_1 . q_2) \\
&= \bar{\gamma}_{\sigma_1} + \bar{\gamma}_{\sigma_2}((q_1, g_1, (q_2 . g_2))
\end{aligned}$$

Corollary: The group homomorphism γ from $H^1(G,\mathrm{Hom}(Q,K))$ to $H^2_{\bar{p}_K}(QXp_Q G,K)$ induces a group homomorphism from the group $\mathrm{Ext}(T(p_K,p_Q))$ to the group $\mathrm{Ext}_{\bar{p}_K}QXp_Q G,K)$ of group extension classes of $QXp_Q G$ by K.

Definitions: A T-kinematical group is any H-extension of the Static group viewed as a split extension of H by an H-enlargement of T by S. An S-kinematical group is an H-extension of the Static group viewed as a split extension of an H enlargement of S by T. Thus the Galilei group is a T-kinematical group and the Carroll group an S-kinematical group.

Theorem (4): There is an $\mathbb{R}$-module isomorphism between the $\mathbb{R}$-module $H_p{}^1(H,\mathrm{Hom}(T,S))$ and the $\mathbb{R}$-module of equivalence classes of T-kinematical groups and an $\mathbb{R}$-isomorphism between the $\mathbb{R}$-module $H^1_{p'}(H,\mathrm{Hom}(S,T))$ and the $\mathbb{R}$-module of equivalence classes of S-kinematical groups.

Proof: That $H_p{}^1(H,\mathrm{Hom}(T,S))$ is an $\mathbb{R}$-module follows from the fact that S is an $\mathbb{R}$-module and H acts $\mathbb{R}$-linearly an S (as a rotation group). Similarly $H^1_{p'}(H,\mathrm{Hom}(S,T))$ is an $\mathbb{R}$-module because $T \cong \mathbb{R}$ and H acts trivially on T. The next follows from proposition (2).

Theorem (5): There is an $\mathbb{R}$-homomorphism between the $\mathbb{R}$-module of equivalence classes of T-kinematical groups and the $\mathbb{R}$-module of equivalence classes of extensions of TXH by S, and an $\mathbb{R}$-homomorphism between the $\mathbb{R}$-module of equivalence classes of S-kinematical groups and the $\mathbb{R}$-module of equivalence classes of central extensions of $SXp_s H$ by T.

Proof: Use proposition (3) and its corollary 11.

Conjecture (1): That the linear space of equivalence classes of T-kinematical groups is 1-dimensional?

Proof: $H_p{}^1(H,\mathrm{Hom}(T,S)) \cong \mathbb{R}$? Suppose $\sigma \in Z_p{}^1(H,\mathrm{Hom}(T,S))$ then for $(v,r) \in H$, $t \in T$

$$\sigma(v,r)(t_1+t_2) = \sigma(v,r)(t_1) + \sigma(v,r)(t_2)$$
$$\sigma(v_1 + r_1 . v_2, r_1 r_2)(t) = \sigma(v_1, r_1)(t) + r_1 . \sigma(v_2, r_2)(t)$$

These identities result in the following

$$\begin{aligned}\sigma(v,r)(t) &= \sigma((v,e)(0,r))(t)\\ &= \sigma(v,e)(t) + \sigma(0,r)(t)\\ &= \sigma((0,r)(r^{-1}.v,e))(t)\\ &= \sigma(0,r)(t) + r.\sigma(r^{-1}v,e)(t)\end{aligned}$$

Consider the function $\psi : 0(3,\mathbb{R}) \to \mathrm{Hom}(T,S)$;

$$\psi(r)(t) = \sigma(0,r)(t) . \psi \in Z_r{}^1(0(3,\mathbb{R}),\mathrm{Hom}(T,S))$$

For

$$\psi(r_1 r_2)(t) = \psi(r_1)(t) + r_1 . \psi(r_2)(t)$$

But

$$Z_n^1(0(3, R), \mathrm{Hom}(T, S)) = B_n^1(0(3, R), \mathrm{Hom}(T, S))$$

For if P is the inversion $x \mapsto -x$, $C(0(3,\mathbb{R})) = \mathbb{Z}_2(P)$ and one can easily show $\psi(r) = \frac{1}{2}(\delta(\psi(P))(r)$ for any 1-cocycle ψ. Such a coboundary yields a coboundary of $Z_p^1(H, \mathrm{Hom}(T, S))$ so we may equate $\psi(0, r)(t) = 0$ for all $r \in 0(3, \mathbb{R})$, $t \in T$. Thus we are left with

$$\sigma(v, r)(t) = \sigma(v, e)(t) = \chi(v)(t)$$

and the function $\chi : H_0 \to \mathrm{Hom}(T, S)$ satisfies

$$\chi(r^{-1} . v)(t) = r . \chi(v)(t) \qquad \text{and} \qquad \chi(v_1 + v_2) = \chi(v_1) + \chi(v_2)$$

The first must imply that $\chi(v)(t) = \lambda_\sigma(t) . v$ for $\lambda_\sigma : \mathbb{R} \to \mathbb{R}$. For we note that $I(\bar{\chi}_t(v)) = I(v)$ for $\bar{\chi}_t(v) = \chi(v)(t)$ and thus $\bar{\chi}_t(v)$ and v are colinear (where I is the isotropy group in $0(3, \mathbb{R})$). λ_σ is a $\mathbb{Z}$-endomorphism of $\mathbb{R}$. We conjecture λ_σ is in fact $\mathbb{R}$ linear which implies $\lambda_\sigma(t) = t\lambda_\sigma(1) = t\sigma^*$ for $\sigma^* \in \mathbb{R}$. If this is true $|\sigma| \mapsto \sigma^*$ will be an $\mathbb{R}$-isomorphism. (That it is a function follows from the above.) Its inverse is defined by $\alpha \mapsto |\sigma_\alpha|$ where for $\alpha \in \mathbb{R}$, $\sigma_\alpha(v, r)(t) = \alpha v t$.

Corollary: If the above conjecture is true the real linear space of equivalence classes of T-kinematical groups is generated by the equivalence class of the Galilei group.

Proof: The Galilei group is the T-kinematical group corresponding to $\alpha = 1$ above.

Conjecture (2): That the linear space of equivalence classes of S-kinematical groups is 1-dimensional over $\mathbb{R}$?

Proof: If $\phi \in Z_p^1(H, \mathrm{Hom}(S, T))$, we must require that

$$\phi(v, r)(x_1 + x_2) = \phi(v, r)(x_1) + \phi(v, r)(x_2)$$
$$\phi((v_1 + r_1 . v_2, r_1 r_2) = \phi(v_1, r_1)(x) + \phi(v_2, r_2)(r^{-1} x)$$

We obtain then the following identities

$$\phi(v, r) = \phi((v, e)(0, r)) = \phi(v, r)(x) = \phi(v, e)(x) + \phi(0, r)(x)$$
$$\phi(v, r) = \phi((0, r)(r^{-1} v, e)) = \phi(v, r)(x) = \phi(0, r)(x) + \phi(r^{-1} v, e)(r^{-1} . x)$$

If $\phi(0, r) \equiv \mu(r)$, we have

$$\mu \in Z_n^1(0, ", R), \mathrm{Hom}(S, T)) = B^1 n(0(3, R), \mathrm{Hom}(S, T))$$

which induces a boundary of the group $B_p^1(H, \mathrm{Hom}(S, T))$ and we may thus equate $\phi(0, r) = 0$ and obtain $\phi(v, r)(x) = \rho(v)(x)$ where $\rho(v)(x) = \phi(v, e)(x)$ and ρ satisfies $\rho(r . v)(r . x) = \rho(v)(x)$ for any rotation r: with

$$\rho \in \mathrm{Hom}(H_0, \mathrm{Hom}(S, T)) \cong \mathrm{Hom}(H_0 \otimes_{\mathbb{Z}} S, T).$$

Thus ρ must be some $\mathbb{Z}$-bilinear function $\mathbb{R}^3 \otimes_{\mathbb{Z}} \mathbb{R}^3 \to \mathbb{R}$. Which is rotation invariant. We conjecture that then $\rho(v,r) = \alpha_\phi(v|r)$ where $(|)$ is the scalar product in $\mathbb{R}^3$ and $\alpha_\phi \in \mathbb{R}$. If this is true $|\phi| \mapsto \alpha_\phi$ is a bijection

$$H^1 p'(H, \mathrm{Hom}(S,T)) \to \mathbb{R}$$

for the correspondence is a function as we saw above and has an inverse $\alpha \mapsto \phi_\alpha$ where $\phi_\alpha(v,r)(x) = \alpha(v|x)$.

Corollary: If the above conjecture is true then the real linear space of equivalence classes of S-kinematical groups is generated by the equivalence class of the Carroll group.

Proof: The Carroll group corresponds to ϕ_1.

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